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Short Communication

The energetics of cylindrical bending waves in a thin plate

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1. Introduction

Vibrational energy flow in an undamped elastic body can arise from both propagating and evanescent wave components. A good example of this is provided by a simple Euler–Bernoulli beam in which four bending waves exist at any specified frequency. Two of these waves correspond to left and right propagating waves, while the other two are evanescent waves, one of which decays exponentially to the left, and the other decays exponentially to the right. When present in isolation, each of the propagating waves transmits energy, while each of the evanescent waves does not. When all four waves are present at once, the energy flow in the system is complicated by the fact that the two evanescent waves can interact to transmit energy, as clearly described by Bobrovnitskii in Ref. [1] and in subsequent discussion [2,3] arising from that reference. More generally, an expression for the interactive energy flow in a pair of evanescent waves was given in Ref. [4] for the case of systems governed by a transfer matrix, which covers the case of plane waves in any homogeneous system. The analysis of Ref. [4] does not apply to cylindrical waves, although the energetics of cylindrical wavefields can be of interest in the study of high frequency vibrations [5,6]. The aim of the present note is to derive an expression for the interactive energy flow associated with evanescent cylindrical bending waves in a flat plate, and to demonstrate important differences between Hankel function and modified Bessel function descriptions of the evanescent waves.

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2. Energy flow in cylindrical bending waves

2.1. Governing equation and complementary functions

When expressed in polar coordinates $(r,\ \theta)$ the equation of motion that governs the harmonic bending vibration of a thin plate has the form [7]

$$\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}+k^2\right)\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}-k^2\right)w=0, \tag{1}$$

where $w(r,\ \theta)$ is the out-of-plane displacement, which is taken to have the time dependency $\exp(\mathrm{i}\omega t)$, where ω is the vibration frequency. The parameter k that appears in Eq. (1) is the wavenumber, and this is given by $k^2=\omega\sqrt{\rho h/D}$, where ρ is the material density, h is the plate thickness, and D is the flexural rigidity. If the solution to Eq. (1) is written in the form

$$w(r,\theta)=w_n(r)\cos n\theta \quad \text{or} \quad w(r,\theta)=w_n(r)\sin n\theta, \tag{2}$$

where n is any integer, then it follows that a set of independent complementary functions can be derived by considering the solution of [7]

$$\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}-\frac{n^2}{r^2}\pm k^2\right)w_n=0. \tag{3}$$

It is well known that the independent solutions of Eq. (3) associated with $+k^2$ are

$$w_n(r)=\begin{cases} J_n(kr) \\ Y_n(kr) \end{cases} \quad \text{or} \quad w_n(r)=\begin{cases} H_n^{(1)}(kr), \\ H_n^{(2)}(kr), \end{cases} \tag{4, 5}$$

where J_n and Y_n are the Bessel functions of order n of the first and second kind, respectively, and $H_n^{(1)}$ and $H_n^{(2)}$ are the Hankel functions of order n of the first and second kind, respectively [8]. Similarly, the independent solutions to Eq. (3) associated with $-k^2$ are

$$w_n(r)=\begin{cases} I_n(kr) \\ K_n(kr) \end{cases} \quad \text{or} \quad w_n(r)=\begin{cases} H_n^{(1)}(\mathrm{i}kr), \\ H_n^{(2)}(\mathrm{i}kr), \end{cases} \tag{6, 7}$$

where I_n and K_n are the modified Bessel functions of order n of the first and second kind, respectively. The functions appearing in Eq. (5) are generally identified physically as propagating cylindrical waves, while those in Eq. (7) are described as evanescent cylindrical waves. This interpretation is consistent with the following behaviour of the Hankel functions for large argument kr [8]:

$$H_n^{(1)}(kr)\to\sqrt{2/(\pi kr)}\mathrm{e}^{\mathrm{i}kr-\mathrm{i}(\pi n/2+\pi/4)}, \tag{8}$$

$$H_n^{(2)}(kr)\to\sqrt{2/(\pi kr)}\mathrm{e}^{-\mathrm{i}kr+\mathrm{i}(\pi n/2+\pi/4)}, \tag{9}$$

$$H_n^{(1)}(\mathrm{i}kr)\to(2/\pi)(-\mathrm{i})^{n+1}K_n(kr)\to\sqrt{2/(\pi\mathrm{i}kr)}\mathrm{e}^{-kr-\mathrm{i}(\pi n/2+\pi/4)}, \tag{10}$$

$$H_n^{(2)}(\mathrm{i}kr)\to 2(\mathrm{i})^n I_n(kr)\to\sqrt{2/(\pi\mathrm{i}kr)}\mathrm{e}^{kr+\mathrm{i}(\pi n/2+\pi/4)}. \tag{11}$$

Noting that since the time dependency $\exp(\mathrm{i}\omega t)$ has been adopted, Eqs. (8) and (9) clearly correspond to inwardly and outwardly propagating waves, respectively, while Eqs. (10) and (11) represent exponential decay and growth. From this point of view it might be expected that each of the solutions appearing in Eq. (5) will propagate vibrational energy, while each of those appearing in Eq. (7) will not; this is certainly true for the analogous case of propagating and evanescent wave components in a bending beam [1]. The validity of this hypothesis is investigated in what follows by considering the energy flow through a circle of radius a, centred at the origin of the polar coordinate system.

2.2. Energy flow through a circular boundary

As pointed out by Bobrovnitskii in Ref. [9], transmission of vibrational energy through any boundary occurs not only due to the action of the out-of-plane shear force S and the bending moment M but also due to the action of the twisting moment T. The instantaneous power per unit length of circumference at any point on the circle is given by the product of the shear force and the out-of-plane velocity, plus the product of the bending moment and the rate of change of slope in the radial direction, plus the product of the twisting moment and the rate of change of slope in the tangential direction. The total time averaged power can thus be written as

$$P = \tfrac{1}{2}\mathrm{Re}\left\{\int_0^{2\pi}\left[(\mathrm{i}\omega w)^* S + \left(\mathrm{i}\omega\frac{\partial w}{\partial r}\right)^* M + \left(\mathrm{i}\omega\frac{\partial w}{a\partial\theta}\right)^* T\right] a\,\mathrm{d}\theta\right\}, \tag{12}$$

where the integration is around the circumference of the circle, and an asterisk denotes the complex conjugate. However, since the integration is around a closed boundary, it is readily shown (by integration by parts) that Eq. (12) can be written as

$$P = \tfrac{1}{2}\mathrm{Re}\left\{\int_0^{2\pi}\left[(\mathrm{i}\omega w)^*\left(S - \frac{\partial T}{a\partial\theta}\right) + (\mathrm{i}\omega w')^* M\right] a\,\mathrm{d}\theta\right\}, \tag{13}$$

where $S - \partial T/a\partial\theta$ is the Kirchhoff effective shear force [9,10], and a dash represents differentiation with respect to r, so that w' is the slope in the radial direction. Using this notation, the effective shear force and the bending moment are given by [7]

$$S - \frac{\partial T}{a\partial\theta} = D\left(w'' + \frac{w'}{r}\right)' + D\frac{\partial^2}{\partial\theta^2}\left[\frac{(2-v)w'}{r^2} - \frac{(3-v)w}{r^3}\right], \tag{14}$$

$$M = -Dw'' - vD\left(\frac{w'}{r} + \frac{1}{r^2}\frac{\partial^2 w}{\partial\theta^2}\right), \tag{15}$$

where v is the Poisson ratio. By adopting Eq. (2) and taking $w_n(r)$ to satisfy Eq. (3), it can be shown that Eqs. (13)–(15) reduce to the following simple expression

$$P = \pm\pi a D\omega k^2 \varepsilon_n \mathrm{Im}[ww'^*], \tag{16}$$

where the $\pm$ sign corresponds to the $\pm$ sign in Eq. (3), and $\varepsilon_n = 1$ unless $n = 0$, in which case $\varepsilon_n = 2$. The result given by Eq. (16) is closely analogous to the energy flow in a membrane (given by the average value of the pre-tension multiplied by the slope and the velocity), and this is consistent with the fact that Eq. (3) is a form of the membrane equation of motion.

Eq. (16) can also be written in the form

$$P = \mp \mathrm{i}\pi a D\omega k^2 \varepsilon_n W\{w, w^*\}/2, \tag{17}$$

where the $\mp$ sign corresponds to the $\pm$ sign in Eq. (16), and $W\{w, w^*\}$ is the Wronskian of the functions w and w^*. The Wronskian $W\{y_1(r), y_2(r)\}$ of two functions $y_1(r)$ and $y_2(r)$ is defined as the determinant $y_1(r)y'_2(r) - y'_1(r)y_2(r)$, where a dash represents differentiation with respect to the argument r (see, for example, Ref. [11]). If the two functions $y_1(r)$ and $y_2(r)$ are linearly independent, then the Wronskian is non-zero—conversely, if the Wronskian is zero, then the two functions are linearly dependent. The energy flow associated with each of the complementary functions given by Eqs. (5)–(7) can now be investigated.

2.3. Energy flow associated with the various complementary functions

2.3.1. Propagating wave components

The following relation holds for Hankel functions of real argument [8]

$$H_n^{(1,2)^*}(kr) = H_n^{(2,1)}(kr), \tag{18}$$

where the notation $H_n^{(1,2)}$ has been used to represent the Hankel function of either the first or the second kind. Considering a general linear combination of the two solutions given in Eq. (5), it follows from Eq. (17) that

$$\begin{aligned} w_n &= \alpha_n H_n^{(1)}(kr) + \beta_n H_n^{(2)}(kr) \\ &\Rightarrow P = -\mathrm{i}\pi a D\omega k^2 \varepsilon_n(\alpha_n^*\alpha_n - \beta_n\beta_n^*) W\{H_n^{(1)}(ka), H_n^{(2)}(ka)\}/2, \end{aligned} \tag{19}$$

where α_n and β_n are arbitrary complex amplitudes. Now formula 9.1.17 of Ref. [8] yields

$$W\{H_n^{(1)}(ka), H_n^{(2)}(ka)\} = -4\mathrm{i}/(\pi a), \tag{20}$$

and so Eq. (19) can be written as

$$w_n = \alpha_n H_n^{(1)}(kr) + \beta_n H_n^{(2)}(kr) \Rightarrow P = -2D\omega k^2 \varepsilon_n(|\alpha_n|^2 - |\beta_n|^2). \tag{21}$$

Clearly the energy flow is independent of the radius a of the circle, as required for power conservation. Also, there is no *interactive* energy flow between the two propagating wave components, i.e. when both waves are present, the total energy flow is equal to the sum of the energy flow in each individual wave component. Furthermore, if one of the two amplitudes is zero, then

$$w_n = H_n^{(1,2)}(kr) \Rightarrow P = \mp 2D\omega k^2 \varepsilon_n, \tag{22}$$

where the $-$ve sign corresponds to $w_n = H_n^{(1)}(kr)$. Here, $H_n^{(1)}(kr)$ is associated with incoming power while $H_n^{(2)}(kr)$is associated with outgoing power, as predicted in Section 2.1. It can be noted that Eq. (22) is identical to the result given by Eqs. (20) and (21) of Ref. [5] once a scaling factor of π has been taken into account.

2.3.2. Evanescent wave components

The following relation holds for the modified Bessel functions that appear in Eq. (6) [8]:

$$b_n^{(1,2)^*}(kr) = b_n^{(1,2)}(kr), \tag{23}$$

where the notation $b_n^{(1,2)}$ has been used to represent the modified Bessel function of order n of either the first or the second kind, i.e. $b_n^{(1)} = I_n$ and $b_n^{(2)} = K_n$. It then follows from Eq. (17) that for any linear combination of the two functions

$$\begin{aligned} w_n &= \alpha_n I_n(kr) + \beta_n K_n(kr) \\ &\Rightarrow P = -\pi a D\omega k^2 \varepsilon_n \mathrm{Im}[\alpha_n \beta_n^*] W\{I_n(kr), K_n(kr)\}, \end{aligned} \tag{24}$$

where again α_n and β_n are arbitrary complex amplitudes. The Wronskian in Eq. (24) is found from formula 9.6.15 of Ref. [8], so that

$$W\{I_n(ka), K_n(ka)\} = -1/a, \tag{25}$$

and hence Eq. (24) can be written as

$$w_n = \alpha_n I_n(kr) + \beta_n K_n(kr) \Rightarrow P = \pi D\omega k^2 \varepsilon_n \mathrm{Im}[\alpha_n \beta_n^*]. \tag{26}$$

Eq. (26) clearly shows that there can be *interactive* energy flow between the two evanescent cylindrical waves represented by the solutions given in Eq. (6). If one of the two amplitudes is zero or if they are both real or both imaginary, or if they are both complex but linearly dependent through a real constant, then the power is zero—otherwise it is non-zero. This confirms that there is no energy flow associated with either of the two solutions appearing in Eq. (6), although they may interact in such a way that energy propagation will occur. It is well known that evanescent waves can interact to propagate energy [1]; the contribution made here is to quantify this effect for cylindrical bending waves, via Eq. (26).

Now, the power flow associated with each of the two Hankel functions in Eq. (7) can be found by noting that

$$2I_n(kr) = (-\mathrm{i})^n (H_n^{(1)}(\mathrm{i}kr) + H_n^{(2)}(\mathrm{i}kr)), \tag{27}$$

$$2K_n(kr) = \pi(\mathrm{i})^{n+1} H_n^{(1)}(\mathrm{i}kr), \tag{28}$$

which can be obtained from formulae 9.6.3 and 9.6.4 of Ref. [8]. It follows immediately that

$$2\pi\mathrm{i} I_n(kr) + (-1)^{n+1} 2K_n(kr) = (-1)^n \pi(\mathrm{i})^{n+1} H_n^{(2)}(\mathrm{i}kr). \tag{29}$$

Eq. (28) states that $H_n^{(1)}(\mathrm{i}kr)$ is either real or imaginary (since $K_n(kr)$ is real when kr is real) and that it can be written in terms of only *one* of the modified Bessel functions. Thus, in Eq. (26), the amplitude α_n is zero and consequently $H_n^{(1)}(\mathrm{i}kr)$ has zero energy flow. On the other hand, Eq. (29) states that $H_n^{(2)}(\mathrm{i}kr)$ is always complex and that it can be written as a linear combination of the modified Bessel functions, where the amplitudes have different moduli and are linearly dependent through an imaginary constant. Hence, from Eq. (26), $H_n^{(2)}(\mathrm{i}kr)$ *does* permit energy flow, and this is given by

$$w_n = H_n^{(2)}(\mathrm{i}kr) \Rightarrow P = (-1)^{n+1} 4D\omega k^2 \varepsilon_n. \tag{30}$$

Clearly the direction of the energy flow depends on the order n of the complementary function. In some ways Eq. (30) is a surprising result given the asymptotic behaviour of $H_n^{(2)}(\mathrm{i}kr)$ represented by Eq. (11). However, although Eq. (11) suggests that $H_n^{(2)}(kr)$ is proportional to $I_n(kr)$ in the far field, Eq. (29) reveals that there is also a small non-zero component proportional to $K_n(kr)$; this component interacts with $I_n(kr)$ to produce an energy flow that is independent of r. Power balance is made possible by the fact that $H_n^{(2)}(\mathrm{i}kr)$ is singular at the origin, so that the origin acts as a source (or sink) of power that is propagated to (or from) infinity.

3. Conclusion

This note has considered the energy flow associated with cylindrical bending waves in a flat plate. The main results are Eq. (19) for propagating waves and Eq. (24) for evanescent waves. It can readily be demonstrated that there is no interactive energy flow between propagating and evanescent waves, and hence these two equations cover all possible mechanisms of energy transfer in a cylindrical wavefield. It has been shown that: (i) if the modified Bessel functions $I_n(kr)$ and $K_n(kr)$ are used to model the evanescent field, then energy flow can only occur through interaction of the evanescent waves; (ii) if the Hankel functions $H_n^{(1)}(\mathrm{i}kr)$ and $H_n^{(2)}(\mathrm{i}kr)$are used to model the evanescent field then, in addition to interactive energy flow, the second function is associated with a direct energy flow via Eq. (30).

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